# UNIT 2

## Moments For Discovery Solutions

This unit will provide solutions and expected outcomes for all the self-discovery units in the text. Many experiments, particularly those involving *Sketchpad*, do not have a unique answer, and there is no way to predict the outcome a student may have for an experiment. Thus, we present only a particular outcome here, one that is very likely to be obtained by a typical student. The *Sketchpad* experiments appearing in the problem sets will be covered in the Problem Solutions (UNIT 3).

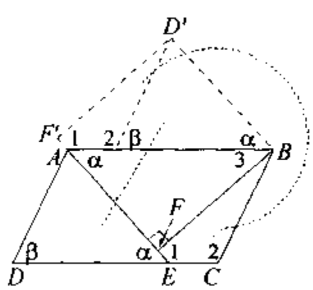
For easier location, the discovery unit solutions are listed by page number rather than by chapter section.

### *Moment for Discovery*, page 4

**Dissection of Parallelogram to Form a Square**

**1–3.** Routine construction.

**4.** See figure. *Proof:* By observing the angle relationships in the figure, $\diamond F'FBD'$ is observed to be a rectangle. To verify it is actually a square, the area of $\square F'FBD'$ = area ▱$ABCD \rightarrow F'F \cdot FB =$ $AE \cdot FB = 12\,FB = 144 \rightarrow FB = 12 = AE = F'F$ and two adjacent sides of $\square F'FBD'$ are congruent. (This construction works for any parallelogram as long as $E$ falls on $\overline{CD}$ and $F$ falls on $\overline{AE}$.)



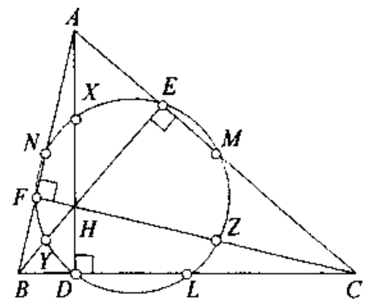

$m\angle 1 + \alpha = 180$

$m\angle 2 + \beta = 180$

$m\angle 3 + \alpha = 90$

## *Moment for Discovery,* page 5

### A Special Circle




**1–2.** Routine construction.
**3.** The angles at $D$, $E$, and $F$ are right angles. Thus $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes of the triangle (see figure).
**4.** $X$, $Y$, and $Z$ lie on the circle.
**5.** ***Conclusion:*** The circle contains the nine points $L$, $M$, $N$, $D$, $E$, $F$, $X$, $Y$, and $Z$.

## *Moment for Discovery,* page 18

### Calculating π

1. 1.414 213 562, 1.847 759 065, 1.961 570 561
2. $2_1$ = 1.414 213 562, $2_2$ = 1.847 759 065, $2_3$ = 1.961 570 561
3. $2_4$ = 1.990 369 453, $2_5$ = 1.997 590 912, $2_6$ = 1.999 397 637, $2_7$ = 1.999 849 404
4. Yes, within 0.00015 units
5. 0.012 271 769
6. 3.141 572 94
7. $2_{10}$ = 1.999 997 647, $\sqrt{2-2_{10}}$ = 0.001 533 981, $2^{11}\cdot\sqrt{2-2_{10}}$ = 3.141 592 35

## *Moment for Discovery,* page 31

### Pedal Triangles and the Simpson Line

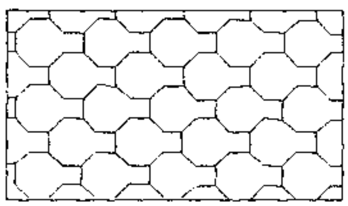
Following the instructions will produce the pedal triangle $\triangle EFG$ of $\triangle ABC$ with respect to point $D$.
***Conclusions:*** (1)There are two positions for $D$ that make $\triangle EFG$ an equilateral triangle, one point inside $\triangle ABC$, the other, outside. (2) If $D = O$ (circumcenter of $\triangle ABC$) then $\triangle EFG \sim \triangle ABC$. (3)If $D$ lies on the circumcircle of $\triangle ABC$, Area $\triangle EFG = 0$ and $E$, $F$, and $G$ are collinear.

# UNIT 3

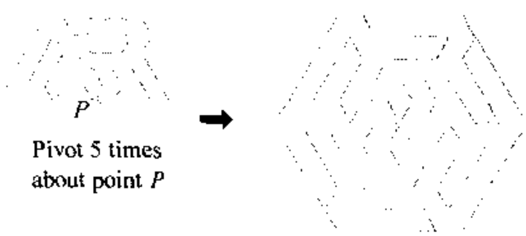
## Solutions to Problems

### Section 1.1: Discovery in Geometry

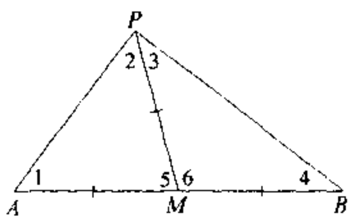
1. Angle sum equals 360.
2. See figure below.



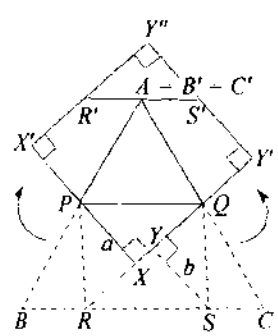
3. Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are tangent to the circle.
4. **Theorem:** Any angle inscribed in a semicircle is a right angle.
5. Isosceles triangle (Side-Splitting Theorem); $\rightarrow$ (implies) $AM = MC = MB$.
6. See figure below.



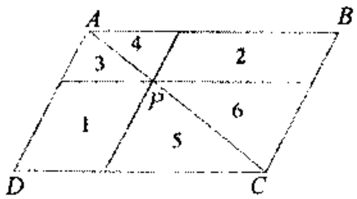

7. **(b)** From (a), $6x + z = 2$, $8x + y + z = 4$, $4x + z = 1$; first and third equations yield $x = \frac{1}{2}$, $z = -1$, and substitution into second equation yields $y = 1 \rightarrow K = \frac{1}{2}B + I - 1$.
8. $MP = MA$ and $MP = MB \rightarrow \triangle APM$ and $\triangle PMB$ are isosceles with angle sum at $P$ = 90: $m\angle 1 + m\angle 2 + m\angle 5 + m\angle 6 + m\angle 3 + m\angle 4 = 360$ or $2m\angle 2 + 2m\angle 3 + 180 = 360$; hence $m\angle 2 + m\angle 3 = 90$ and $\angle APB$ is a right angle.




9. Actual proof that reassembled figure is a square: Pivoting about $P$, $Q$, and $S$ yields closed figure because $P$ and $Q$ are midpoints and $RS = 5 = BC - RS = BR + SC$, or $RS = AR' + AC' = R'S'$. Therefore $\Diamond XY'Y''X'$ is a rectangle; it will be shown that $X'X = XY'$. Area $(XY'Y''X')$ = Area of $\Delta ABC \rightarrow X'X \cdot XY' = (\sqrt{3}/4) \cdot 10^2 = 25\sqrt{3} = QR^2 \rightarrow X'X \cdot XY' = QR^2$. In original figure for problem, $PQ = RS \rightarrow \Diamond PQSR$ is a parallelogram; by congruent triangles, $a = b$ and $RX = QY$. $\therefore XY' = XQ + QY = XQ + RX = QR$ and $X'X \cdot XY' = QR^2 = X'X \cdot QR \rightarrow X'X = QR = XY'$.

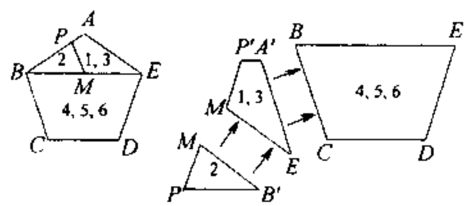


**10.** $\triangle ACD \cong \triangle ABC$ so Area of Region (1, 3, 5) = Area of Region (2, 4, 6). Region 3 ≅ Region 4 and Region 5 ≅ Region 6, so Area (Region 3) = Area (Region 4) and Area (Region 5) = Area (Region 6) → Area (Region 1) = Area (Region 2).




**11. (b)** If $m\angle A = 120°$ in $\triangle ABC$, draw lines $\overline{AD}$ and $\overline{AE}$ forming angles of measure 30°, 60°, and 30° at $A$, then subdivide $\triangle ADE$ by joining vertices to centroid.

**12.** Angles of regular pentagon have measure 108. Horizontal diagonal ($\overline{BE}$ in figure) is parallel to base of pentagon → $m\angle CBE$ = 180 – 108 = 72; $m\angle ABE$ = 108 – 72 = 36.




By construction, $m\angle BMP$ = 72 → $m\angle BPM$ = 180 – 72 – 36 = 72 → $BP = BM$. Region (1,3) fits Region (4, 5, 6) because $AE = BC$ = side of pentagon; $P'$, $A' = B$, and $E$ collinear because $m\angle P'A'E + m\angle CBE$ = 108 + 72 = 180. Region 2 ($\triangle BPM$) fits Region (1, 3) because $BM = ME$ by construction; $P$, $M$ and $P'$ collinear because $m\angle PMB' + m\angle EMP'$ = 72 + (180 – 72) = 180. Finally, $P$, $B'$, and $D$ are collinear because $m\angle B' + m\angle E + m\angle C$ = 36 + 36 + 108 = 180. Resulting quadrilateral is parallelogram because $m\angle P' + m\angle E$ = (180 – 72) + 72 = 180; final dissection from parallelogram to square as introduced previously may be applied to define remaining three pieces.

**13.** $MN = \frac{1}{2}BC = YZ$ and $\overline{MZ} \parallel \overline{AD} \perp \overline{BC} \parallel \overline{ZY}$ (Midpoint Connector Theorem) → $MNYZ$ is rectangle; midpoint $U$ of hypotenuse $\overline{NZ}$ of right triangle $\triangle NZY$ equidistant from $N$, $Y$, $Z$, $M$ (Problem 5) → circle centered at $U$ = midpoint of $\overline{XL}$ = $U'$ passes through $M$, $N$, $Y$, $Z$, $L$, $X$. Finally, $U'$ (= $U$) is equidistant from vertices of right triangle $\triangle XDL$, so circle also passes through $D$, $E$, and $F$.

**14.** $AM = \frac{1}{2} AB = \frac{1}{2}\sqrt{2^2 + 2^2} = \frac{1}{2}\sqrt{8} = \sqrt{2}$ (Pythagorean Theorem); $AQ/AC$ = $\sin 30°/\sin 135° = \frac{1}{2}/(\sqrt{2}/2 = 1/\sqrt{2}$ → $AQ = 2/\sqrt{2} = \sqrt{2} = AM$. $AR = AR$ and $\angle MAR \cong \angle RAQ \to \triangle AMR \cong \triangle AQR$ → $\angle AQR$ is right angle. $m\angle RQP$ = 360 – $m\angle RQA - m\angle AQC - m\angle PQC$ = 360 – 90 – 135 – ½(180 – 30) = 60 = $m\angle RPQ$ (by symmetry) → $\triangle PQR$ is equiangular, ∴ equilateral.

**15. (a)** Each angle of polygon = $128\,\frac{4}{7}°$ → since Region (1, 2, 3) is isosceles triangle, $m\angle BAC = \frac{1}{2}(180° - 128\,\frac{4}{7}°) = 25\,\frac{5}{7}°$, $m\angle ABC = m\angle ACB = 77\,\frac{1}{7}°$; $m\angle DAL$ = $128\,\frac{4}{7}° - 2m\angle BAC = 77\,\frac{1}{7}°$ and $\overline{FN} \parallel \overline{AL}$ → $m\angle ENF = m\angle MAL$ → isosceles triangles are similar, as stated. Angle measures of $\triangle 3$ are $25\,\frac{5}{7}°$ (at angle opposite $B$ and $C$), $51\,\frac{3}{7}°$ (at $B$), $102\,\frac{6}{7}°$ (at $C$); angle between horizontal diagonal and side bordering pieces 6 and 7 = $102\,\frac{6}{7}°$ → $\triangle(1, 2)$ and $\triangle 3$ will fit into spaces shown (adjacent to dashed lines). $m\angle DAL = 77\,\frac{1}{7}°$ → $\triangle 5$ will fit in space shown, and $\overline{LN} \parallel$ dashed line opposite, forming smaller parallelogram → pieces (4, 9), (10, 11), 12 make up rest of larger parallelogram. Remainder of dissection follows from dissection of parallelogram into square.

## Section 1.2: Variations on Two Familiar Geometric Themes

**2.** Angle between two hypotenuses has measure 90 because the acute angles of a right triangle are complementary. Thus, area of half-square = $\frac{1}{2} c^2$; area of trapezoid = sum of three areas: $\frac{1}{2} c^2 + \frac{1}{2} ab + \frac{1}{2} ab = \frac{1}{2} c^2 + ab$. Using formula, area of trapezoid $= \frac{1}{2} (b + a)(b + a) = \frac{1}{2} a^2 + ab + \frac{1}{2} b^2 = \frac{1}{2} c^2 + ab$. $\therefore \frac{1}{2} a^2 + \frac{1}{2} b^2 = \frac{1}{2} c^2 \to a^2 + b^2 = c^2$.

**3.** $x^2$ = area of square $= 4 \cdot 6 + 1^2 = 25 \to x = 5$.

**4.** **(a)** $(1/4)^2 + (1/3)^2 = 25/144 = (5/12)^2$.

**(b)** By area formula for right triangle, $\frac{1}{2} ch_3 = K \to h_3 = 2K/c = ab/c \to (1/h_1)^2 + (1/h_2)^2 = (1/a)^2 + (1/b)^2 = (a^2 + b^2)/a^2b^2$ (algebra) $= c^2/a^2b^2 = (1/h_3)^2$.

**5.** **(a)** 3.132 628 6;

**(b)** 3.139 350 2;

**(c)** 3.141 557 6.

**6.** When $0 < x_1 < 2\pi$, sequence converges to $\pi$. If $x_1 = \pi$ or $2\pi$, sequence is constant ($= \pi$); if $x_1 > 2\pi$, sequence converges to odd multiple of $\pi$ ($3\pi$ if $x_1 = 7$, for example, and $5\pi$ if $x_1 = 13$, etc.) [*NOTE for more advanced readers:* A series expansion for $\sin(\pi + \varepsilon) = -\sin \varepsilon$ shows that if $P = \pi + \varepsilon$, $P + \sin P = \pi + \varepsilon - \sin \varepsilon \approx \pi + \varepsilon^3/6$, so if $P \approx \pi$ to one decimal of accuracy, $P + \sin P \approx \pi$ to 3 decimals, and accuracy increases by at least 3 decimal places each iteration.]

**8.** (1) Measure of angle of ▱ at $C = 270° - C \to$ Area $= ab \sin(270° - C) = -ab \cos C$.

(2) From the shaded region (to left), cut off a triangle congruent to $\triangle ABC$ at top and right, and fit these into spaces inside the square at left and bottom $\to$ area of shaded region = area of square.

(3) Area (square) $= c^2$ = Area (square of side $a$) + Area (square of side $b$) + $2 \cdot$Area (▱) $= a^2 + b^2 - 2ab \cos C$.

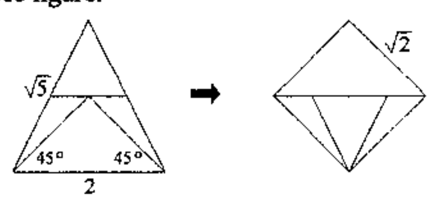
**9.** Regular inscribed hexagon has side $s_2 = 1$, so regular dodecagon has side $s_3 = \sqrt{2 - \sqrt{4 - 1^2}} = \sqrt{2 - \sqrt{3}}$; $s_4$ (24 sides) $= \sqrt{2 - \sqrt{4 - s_3^2}} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{3})}} = \sqrt{2 - \sqrt{2 + \sqrt{3}}}$, and $s_5$ (48 sides) $= \sqrt{2 - \sqrt{4 - s_4^2}} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{2 + \sqrt{3}})}} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}}$. In general, $s_n$ ($3 \cdot 2^n$ sides) $= \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}}$ ($n$ radicals); multiply this by $\frac{1}{2} (3 \cdot 2^n)$ for $\pi'_n$.

**10.** $n = 6$ corresponds to regular 96-sided polygon; from **(6)** we obtain

$$\pi_6 = 3 \cdot 2^5 \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}} = 3.141\,452\,5$$

**11.** (Refer to construction discussed in discovery unit in Section 1.1.) Radius of construction arc $= \sqrt{bh} = \sqrt{8 \cdot 4} = \sqrt{32} \approx 5.7$, and arc meets opposite side at $E$ where $DE = 7$, $EC = 1$. By measuring $\angle AEB$ with a protractor (or other means of analysis), $m\angle AEB = 90° \to F = E \to$ construction in Section 1.1 valid (with $F = E$).

**12.** See figure.




**13.** **(a)** $AE = AM = AM + MC = a/2 + \sqrt{a^2 + (a/2)^2} = a/2 + (a/2)\sqrt{5} = \tau a$, so $AE/AD = \tau$.

**(b)** Quadratic Formula gives $x = \frac{1}{2} \cdot [1 \pm \sqrt{1^2 + 4}] = \frac{1}{2} (1 + \sqrt{5}) = \tau$

**(c)** $\tau^2 = \tau + 1 \to \tau^2/\tau = (\tau + 1)/\tau = 1 + \tau^{-1} \to \tau^{-1} = \tau - 1$. Next, $\tau^{-2} = (\tau - 1)^2 = \tau^2 - 2\tau + 1 = (\tau + 1) - 2\tau + 1 = 2 - \tau$.

**14.** **(a)** Without loss of generality it may be assumed that $BC = 1 \to AB = \tau \to EB = AB - AE = AB - BC = \tau - 1 = 1/\tau$ [$\tau^2/\tau = (\tau + 1)/\tau \to \tau = 1 + 1/\tau$]. $\therefore EB = 1/\tau$ and $BC/BE = 1/(1/\tau) = \tau \to$ remaining rectangles in figure for problem are Golden Rectangles.

**(b)** Sides of squares, beginning with $ADFE$ and continuing in the counterclockwise direction, have respective lengths 1 ($= AE$),

$\tau^{-1}$ (= $EB$), $\tau^{-2}$, $\tau^{-3}$, $\cdots$ so their areas are the squares of these numbers, or, 1, $\tau^{-2}$, $\tau^{-4}$, $\tau^{-6}$, $\cdots$.

**(c)** $\tau$ = Area (□$ABCD$) = sum of areas of all subsquares $= 1 + \tau^{-2} + \tau^{-4} + \tau^{-6} + \cdots$.

**15.** Rigorous induction proof of Viète's expression for $\pi$, can be formulated as follows (making use of the notation $2_n$ introduced in the discovery unit of this section): From **(5')** $2^n\sqrt{2 - 2_{n-1}} \to \pi$ as $n \to \infty$, so $2^k\sqrt{2 - 2_k} \to \pi/2$ as $k \to \infty$. Observe that $1/2_1 = 1/\sqrt{2}$, $1/2_2 = 1/\sqrt{2 + 2_1} = \sqrt{2 - 2_1}/\sqrt{2}$ (algebra). For $n \geq 2$,

$$\frac{1}{2_{n+1}} = \frac{1}{\sqrt{2+2_n}} = \frac{1}{\sqrt{2+2_n}} \cdot \frac{\sqrt{2-2_n}}{\sqrt{2-2_n}} =$$

$$\frac{\sqrt{2-2_n}}{\sqrt{4-2_n^{\,2}}} = \frac{\sqrt{2-2_n}}{\sqrt{4-(2+2_{n-1})}} = \frac{\sqrt{2-2_n}}{\sqrt{2-2_{n-1}}}$$

$$\therefore\ \frac{1}{2_1}\cdot\frac{1}{2_2}\cdot\frac{1}{2_3}\cdots\frac{1}{2_{n+1}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2-2_1}}{\sqrt{2}}\cdot$$

$$\frac{\sqrt{2-2_2}}{\sqrt{2-2_1}}\cdot\frac{\sqrt{2-2_3}}{\sqrt{2-2_2}}\cdots\frac{\sqrt{2-2_n}}{\sqrt{2-2_{n-1}}} =$$

$\frac{1}{2}\sqrt{2-2_n}$. $\therefore$ $(2/2_1)(2/2_2)(2/2_3)\cdots(2/2_{n+1}) = 2^n\sqrt{2-2_n} \to \pi/2$ as $n \to \infty$. Thus,

$$\frac{2}{2_1}\cdot\frac{2}{2_2}\cdot\frac{2}{2_3}\cdots\frac{2}{2_{n+1}} \to \frac{\pi}{2} \text{ and therefore}$$

$$(*)\qquad \frac{2_1}{2}\cdot\frac{2_2}{2}\cdot\frac{2_3}{2}\cdots\frac{2_{n+1}}{2}\cdots = \frac{2}{\pi}$$

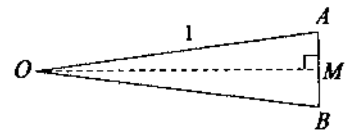
Define the symbols $(\frac{1}{2})_1 = \sqrt{\frac{1}{2}}$, $(\frac{1}{2})_2 = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}$, ... , $(\frac{1}{2})_{n+1} = \sqrt{\frac{1}{2} + \frac{1}{2}(\frac{1}{2})_n}$. Viète's expression then becomes $2/\pi = (\frac{1}{2})_1\cdot(\frac{1}{2})_2\cdot(\frac{1}{2})_3\cdots(\frac{1}{2})_{n+1}\cdots$. In view of **(*)** it remains to show that $2_n/2 = (\frac{1}{2})_n$ for all $n$ (true for $n = 1$ since $\sqrt{2}/2 = \sqrt{\frac{1}{2}}$. Suppose $2_{n-1}/2 = (\frac{1}{2})_{n-1}$ for some $n > 1$; then

$$\frac{2_n}{2} = \frac{\sqrt{2+2_{n-1}}}{2} = \sqrt{\frac{2+2_{n-1}}{4}} =$$

$$\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cdot 2_{n-1}/2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}(\tfrac{1}{2})_{n-1}} = (\tfrac{1}{2})_n.$$

**16.** Central angle $\angle AOB$ of regular $2^{n-1}$-sided regular polygon having unit radius has measure $90°/2^{n-1}$ so in figure, using $\triangle AOM$, $\sin \frac{1}{2}(90°/2^{n-1}) = MA/OA = \frac{1}{2}s_n$ $\to \sin 90°/2^n =$

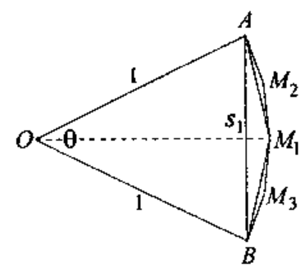
$$\tfrac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{2 + \sqrt{2}}}}}\ \text{[by (4)]}.$$




**17.** If $a = 2.5$, limit $\approx 3.295\,291 \neq \pi$. Problem 18 addresses the issue in detail.

**18.** Converges to $4\sin^{-1}(\frac{1}{2}\sqrt{4-a})$, as the following analysis shows (see figure): $s_2 = AM_1 = \sqrt{2 - \sqrt{4 - s_1^{\,2}}} = \sqrt{2 - \sqrt{a}}$, and $s_3 = AM_2 = \sqrt{2 - \sqrt{4 - s_2^{\,2}}} = \sqrt{2 - \sqrt{4 - (2 - \sqrt{a})}}$ $= \sqrt{2 - \sqrt{2 + \sqrt{a}}}$ [a sequence of steps that has the obvious generalization parallel to text derivation of **(4)**]. Length of arc $\widehat{AB}$ is approximated by $2\cdot AM_1$, $4\cdot AM_2$, $\cdots$, so exact length is the limit, as $n \to \infty$, of

$$2^{n-1}\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{2 + \sqrt{a}}}}} = s.$$ Using the relation $s = r\theta$ for the same arc length, and trigonometry, $\sin \frac{1}{2}\theta = \frac{1}{2}AB/AO = \frac{1}{2}s_1$ or $\sin s/2 = \frac{1}{2}\sqrt{4-a} \to s = 2\sin^{-1}\frac{1}{2}\sqrt{4-a}$.

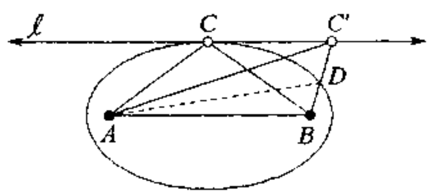



If $a = 4$, members of sequence = 0; if $a >$ 4, $\sqrt{2 + \sqrt{2 + \cdots \sqrt{2 + \sqrt{a}}}} > 2$ and sequence diverges to $\infty$. (*NOTE:* This result yields exact answer to sequence in Problem 17, $2a\sin^{-1}\frac{1}{2}\sqrt{4-a}$. It can be shown that this has the value $\pi$ *only* when $a = 2, 3$).

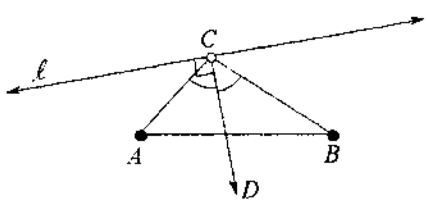
**19.** By similar triangles, $GB/BE = FB/BD = (BE/BD)/BD \to GB = BE^2/BD^2 = (^1/_2)^2 \cdot [(^7/_8)^2 + 1] = {}^{16}/_{113} \to CL = 3 + {}^{16}/_{113} = {}^{355}/_{113}$.

## Section 1.3: Discovery via the Computer

1. Perimeter unbounded; area constant because altitude and base of $\Delta ABC$ invariant (perimeter minimum when $C$ is directly above midpoint of $\overline{AB}$, that is, when $\triangle ABC$ is isosceles). Proof of this last part can be had by drawing an ellipse with foci at $A$ and $B$ passing through $C$ (figure). If $C'$ is any other point on $\ell$ and $D$ is point of intersection of $\overline{BC}$ with ellipse, then $AC' + C'B = AC' + C'D + DB > AD + DB = AC + CB$.




2. Area and perimeter both unbounded; minimum area = 0 when $C$ = intersection of $\ell$ and line $\overleftrightarrow{AB}$. Minimum perimeter occurs when $AC + CB$ is least, which occurs when $m\angle ACD = m\angle DCB$, where $\overleftrightarrow{CD}$ is line perpendicular to $\ell$. (See Section 5.2; this same construction applies to Problem 1 Solution.)




3. $G$ invariant = midpoint of arc (*Proof*: If $m\widehat{CG} = m\widehat{GF}$, ½ $m\angle CDG = m\widehat{CG} = m\widehat{GF}$ = ½ $m\angle GDF$ so $G$ lies on the bisector of $\angle CDF$ for all points $D$ on arc $\widehat{CF}$.)
4. Observable theorem: *The medians of a triangle are concurrent.*
5. $x \geq 0$ with = only when $D$ lies on minor arc $\widehat{AC}$; $x$ increases to a maximum at point $B$ as $D$ varies on arc $\widehat{ABC}$. *Ptolemy's Theorem*: If $ABCD$ is inscribed quadrilateral, $ac + bd = mn$, where $m$, $n$ are lengths of diagonals.
6. **(a)** The value $x + y + z$ is constant = $\sqrt{3}a$, where $a = BC$.
   **(b)** $x + y + z > \sqrt{3}a$ and has its minimum when $P$ lies on one of the sides of $\triangle ABC$; there is no maximum value.
7. **(a)** Value of $x + y + z$ repeats six times for one revolution of $P$ about circle, with maximum at the three points of tangency, minimum at points half-way between.
   **(b)** $x^2 + y^2 + z^2 = 5a^2/4$ (constant) where $a$ = side of equilateral triangle.
8. Result of experiment is Morley's triangle ($\triangle DEF$), which is always equilateral, independent of the shape of $\triangle ABC$.
9. $\overleftrightarrow{AT}$, $\overleftrightarrow{BU}$, $\overleftrightarrow{CV}$ are perpendicular bisectors to sides of $\Delta PQR$ so meet at circumcenter (point equidistant from vertices). ∴ altitudes of $\Delta ABC$ (same three perpendiculars) are concurrent.
11. Maximum area when $C$ is above midpoint of $\overline{AD}$ (where altitude is maximum); $AE + ED = AB$ = constant → perimeter constant.
12. **(a)** , **(b)** Answer affirmative for both.
    **(c)** *Theorem:* The circumcenter $O$, orthocenter $H$, and centroid $G$ of any triangle are collinear, with $G$ the "two-thirds" point on segment $\overline{HO}$ from $H$ to $O$ ($OH = 3\cdot OG$).
13. $U$ lies on Euler Line; it is midpoint of $\overline{HO}$ and $HG = 2GO$.
14. Line $\overleftrightarrow{TV}$ always passes through point $F$, yielding a harmonic configuration for $H$, $R$, $G$ and $F$.
15. $\Delta KLM$ is similar to original triangle $\Delta ABC$.